PHYSICAL REVIEW E **68**, 026208 (2003)

Isospectrality in chaotic billiards

Abhishek Dhar,[1,2] D. Madhusudhana Rao,[1] Udaya Shankar N.,[1] and S. Sridhar[1,3]
[1]*Raman Research Institute, Bangalore 560080, India*
[2]*Physics Department, University of California, Santa Cruz, California 95064, USA*
[3]*Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA*
(Received 23 March 2002; published 18 August 2003)

We consider a modification of isospectral cavities whereby the classical dynamics changes from pseudointegrable to chaotic. We construct an example where we can prove that isospectrality is retained. We then demonstrate this explicitly in microwave resonators.

DOI: 10.1103/PhysRevE.68.026208 PACS number(s): 05.45.Ac, 03.65.Ge, 41.20.−q

Recently, it has been shown that it is possible to construct two drums which have different shapes but sound exactly the same [1]. This answers the famous question asked by Kac [2]: "Can you hear the shape of a drum?," the answer being "no," Gordon, *et al.* constructed an example of a pair of two-dimensional (2D) domains which had different shapes but had identical eigenvalue spectra for the Laplace operator [3,4]. Since then, a large number of such isospectral pairs have been obtained.

One common feature of all shapes constructed so far is that they are mostly polygonal. Hence, the classical dynamics of a particle in billiards of these shapes is pseudointegrable. A question of interest then is whether isospectrality can be achieved even for cavities with chaotic dynamics, which is typical of domains that have convex pieces, and hence are nonpolygonal. We address this question, viz. are there sound-alike chaotic drums, both theoretically and through experiments using microwave resonators.

Isospectrality is fundamentally a consequence of topology. The essential aspects of isospectrality can be proved using the example of the two isospectral domains $C1$ and $C2$ shown in Fig. 1. The proof consists in showing that given any eigenfunction in one domain, we can construct a corresponding one in the other domain with the same eigenvalue and vice versa. Each domain consists of seven distinct subdomains, each in the form of a triangle. We label these subdomains in an arbitrary fashion, using numbers 1,2, . . . ,7 for domain $C1$ and the alphabets $A,B, \dots ,G$ for domain $C2$. Note that the edges of the triangles are marked differently (by dotted, dashed, and solid lines), and this allow us to make a unique correspondence between any pairs of triangles. Consider any wave function ψ in domain $C1$, which satisfies the eigenvalue equation $-\nabla^2\psi = k^2\psi$ with Dirichlet boundary conditions (ψ vanishes on the boundary of the domain). Let us denote by ψ_i the restriction of the wave function ψ in subdomain i [i.e., $\psi_i(\overline{r}) = \psi(\overline{r})$, if $\overline{r}$ is a point in the ith subdomain, else $\psi_i(\overline{r}) = 0$]. Similarly, we can define the restricted wave functions $\{\psi_A,\ \psi_B, \dots, \psi_G\}$ from any wave function in domain $C2$. Starting from the wave function ψ in $C1$, let us construct the following restricted wave functions in domain $C2$:

$$
\begin{aligned}
\psi_A &= \psi_2 - \tilde{\psi}_1 + \psi_7,\\
\psi_B &= \psi_3 + \psi_1 + \psi_5,\\
\psi_C &= -\tilde{\psi}_3 + \psi_2 + \psi_4,\\
\psi_D &= \psi_4 - \tilde{\psi}_1 + \psi_6,\\
\psi_E &= \psi_5 - \tilde{\psi}_2 - \tilde{\psi}_6,\\
\psi_F &= \psi_7 - \tilde{\psi}_3 + \psi_6,\\
\psi_G &= -\tilde{\psi}_7 + \psi_5 - \tilde{\psi}_4.
\end{aligned}
\tag{1}
$$

The notation used requires some explanation: To construct $\psi_A = \psi_2 - \tilde{\psi}_1 + \psi_7$, we first move the three domains 1, 2, and 7 so that they are on top of each other and all similarly marked edges coincide. This may require us to flip the domains about one of the bases, and in such cases, we have denoted the wave function with a tilde (e.g., $\tilde{\psi}_1$). The wave function ψ_A is then obtained by adding (or subtracting) the values of the three functions at each point. It is easy to see that $\psi' = \psi_A + \psi_B + \psi_C + \psi_D + \psi_E + \psi_F + \psi_G$ is an eigenfunction for domain $C2$ with the same eigenvalue. For this, we notice that (1) Laplace's equation is satisfied in every domain, and (2) it can be verified that the wave function vanishes on the boundary and matches smoothly across subdomains. For example, consider the subdomains A and B that are separated by a dashed line. The wave functions are given by $\psi_2 - \tilde{\psi}_1 + \psi_7$ and $\psi_B = \psi_3 + \psi_1 + \psi_5$. The smoothness follows since from the wave function in $C1$, we see that ψ_2

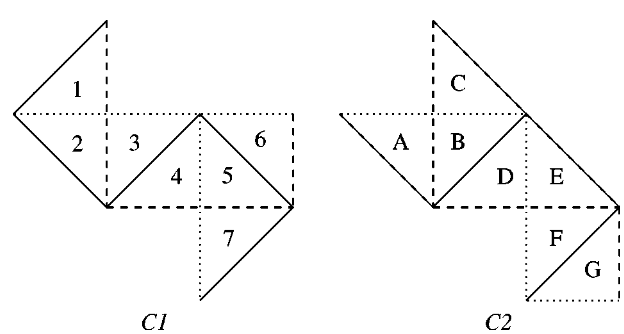

FIG. 1. Isospectral cavities $C1$ and $C2$. The outer edges of the polygonal structure constitute the boundary of the cavity. The inner edges have been marked to show the seven triangular subdomains within each cavity.

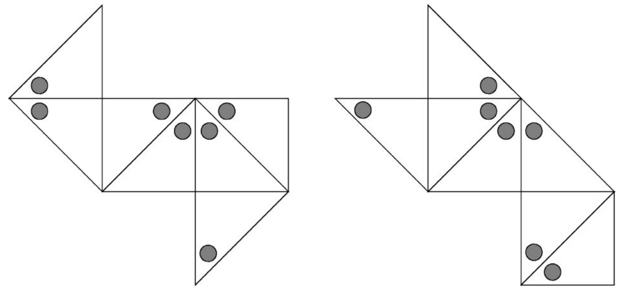

FIG. 2. Isospectral cavities with scatterers in the shape of disks. The wave function vanishes on the boundary and inside of every scatterer.

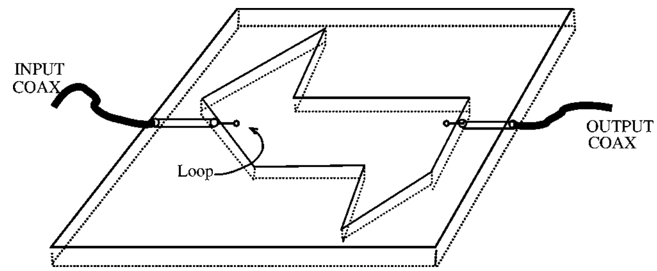

FIG. 3. A schematic of the experimental cavity. This shows a brass plate in which a hole of the desired cavity shape has been cut. This plate is sandwiched between two other brass plates to form a closed cavity.

matches smoothly with ψ_3 across the dashed boundary, similarly ψ_7 matches with ψ_5 and $-\tilde{\psi}_1$ matches with ψ_1.

Similarly, one can construct an eigenfunction for $C1$ starting from any given eigenfunction in $C2$. Thus, we have demonstrated a one-to-one correspondence between the states in the two cavities and hence proved isospectrality.

We now modify the domain geometry so as to make the dynamics chaotic. It is expected that making a part of the boundary convex (inwards into the domain) should make the dynamics chaotic. This is related to the fact that on such boundaries, any two particle trajectories which are close to each other diverge rapidly after being reflected [5]. A well known example of a chaotic billiard is the Sinai billiard obtained by placing a circular scatterer inside a square. In our case, to obtain the modified geometry, we first place a scatterer of arbitrary shape inside one of the triangular subdomains of any one domain and then place one in a similar position in every other triangle. An example with disc-shaped scatterers is shown in Fig. 2. The identification of the edges on the two domains makes this construction unique. Thus, note that in every triangle, the scatterer is placed close to a vertex where a solid and dotted edge meet. The wave function in each domain now changes, since it has to vanish on and inside the boundary of the scatterers. Our construction of the modified geometry with scatterers is such that the proof for isospectrality given above can be repeated, since the wave functions still satisfy the relations given by Eq. (1).

A direct physical proof of isospectrality can be obtained by experiments utilizing microwave cavities [6,7], which provide a simple and powerful method of simulating single-particle time-independent quantum mechanics in two dimensions. This follows from the fact that under appropriate geometrical constraints, Maxwell's equations in a cavity reduces to the Schrödinger equation of a free particle inside a two-dimensional domain of arbitrary shape and topology. In fact, one can show that for a cavity with small thickness (in the z direction, say) compared to the dimensions in the other transverse directions (in the x-y plane), the z-component of the microwave electric field $\Psi(x,y)=E_z$ satisfies the time-independent Schrödinger wave equation $-(\partial_x^2+\partial_y^2)\Psi=k^2\Psi$ (with the identification $k=2\pi f/c$, f being the frequency and c the speed of light) and ψ vanishes on the boundary of the domain. This correspondence is exact for all frequencies $f<c/2d$, where d is the thickness of the cavity. We note that this is also the Helmholtz equation which describes, for example, vibrations of a drum. Using this equivalence, various phenomena (such as quantum chaos) have been studied. Isospectrality has earlier been demonstrated by Sridhar and Kudrolli [8] using microwave cavities shaped as in Fig. 1. In the experiments, one obtains the resonance modes of the cavities. Thus, the microwave transmission spectra directly yields the eigenvalues of the cavity being measured. The advantage of this approach is that it can be easily applied to arbitrary 2D domains for which numerical simulations are very hard [9,10] and may sometimes be practically impossible.

In our experiments, we consider the same set of cavities (Fig. 1) as the ones considered by Sridhar and Kudrolli [8] and investigate the question of isospectrality in the presence of scatterers placed in the specified way inside the cavities. A schematic of the experimental cavity is shown in Fig. 3. The desired domain is cut out from a brass plate of thickness $d=6$ mm. Two other brass plates are placed on top and below the hole to form a closed cavity. As shown in the figure, microwaves were coupled in and out using loops terminating coaxial lines that enter through the sides of the cavity. The length of bases of the triangular subdomains was taken to be $a=8$ cm and the thickness of the cavity was $d=6$ mm. The small thickness of the cavity makes it essentially two-dimensional and the correspondence between Maxwell's equations and Schrödinger's equation is good for frequencies

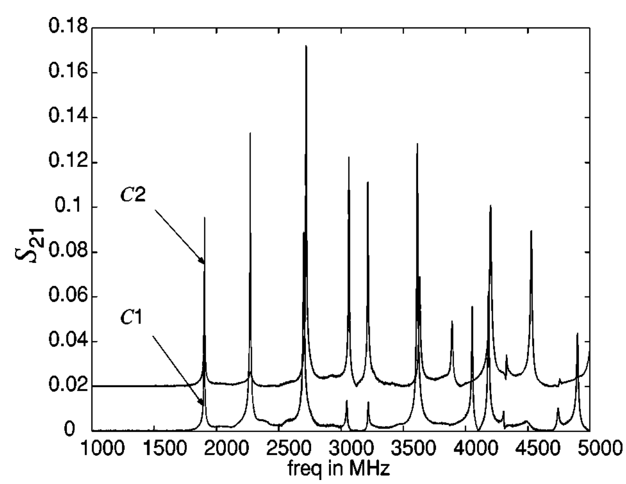

FIG. 4. Comparision of spectrum of isospectral cavities $C1$ and $C2$ in the absence of scatterers. S_{21} is the transmission amplitude.

TABLE I. The first 33 resonances in the two cavities.

Resonant frequency in $C1$ (MHz)	Resonant frequency in $C2$ (MHz)	Percentage discrepancy
1902.500	1903.750	0.0657
2271.250	2274.375	0.1374
2700.625	2719.375	0.6895
3045.625	3062.500	0.5510
3217.500	3215.000	−0.0778
3612.500	3631.250	0.5163
	3892.500	
4054.375		
4184.375	4200.625	0.3868
4303.125	4328.125	0.5776
4488.125	4528.125	0.8834
4743.750	4756.250	0.2628
4898.750		
5026.875	5031.875	0.0994
5171.875	5194.325	0.4322
5426.250	5474.325	0.8782
5488.750		
5625.625	5627.500	0.0333
5793.750	5808.750	0.2582
	5903.125	
5928.750	5940.625	0.1999
6085		
6222.500	6242.500	0.3204
6312.500	6352.500	0.6297
6497.500	6492.500	−0.0770
6680.000	6707.500	0.4100
6750.000	6775.000	0.3690
	6790.000	
6855.000	6877.500	0.3272
6930.000	6992.500	0.8938

$f < c/2d = 25$ GHz. For all metallic objects in the 2D space between the plates, Dirichlet boundary conditions apply inside the metal.

All measurements were carried out using an HP8510B vector network analyzer which measured the transmission (S_{21}) parameters. The typical values of quality factor obtained range from a maximum of 850 at the lower end of the spectrum to a minimum of 250.

TABLE II. The table gives a comparison of the cumulative frequency in the cavities with the Weyl estimate.

Frequency (GHz)	Cumulative resonant frequency	Weyl estimate
1	0	0
2	1	0.8
3	3	3.4
4	7	7.5
5	13	13.1
6	21	20.4
7	30	29.2
8	38	39.5

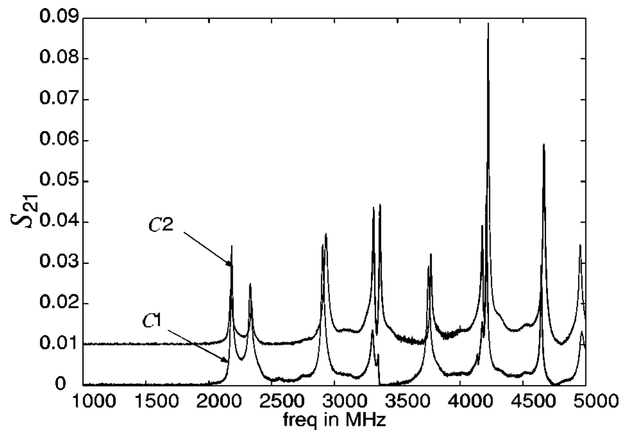

FIG. 5. Comparision of spectrum of the isospectral cavities with scatterers placed as in Fig. 2.

Results. We first attempt to reproduce the results in Ref. [8] for the cavities shown in Fig. 4. We show in Fig. 4 the traces of the spectrum for the two cavities in the frequency range 1-5 GHz. The first 30 resonances of the two cavities are listed in Table I and one sees that the eigenvalues match to better than 1%. One sees that *each resonance present in one is present in the other*. A few lines are missing and this is attributed to the fact that the particular coupling positions we used may not excite some modes. The remaining inaccuracies are due to imperfections in the machining and in the clamping together of various parts of the cavity. Note that the

TABLE III. The first 22 resonant frequencies in the cavities with scatterers.

Resonant frequency in $C1$ (MHz)	Resonant frequency in $C2$ (MHz)	Percentage discrepancy
2181.875	2175.000	−0.3161
2330.000	2338.875	0.3795
2906.250	2933.125	0.9163
3303.750	3311.250	0.2265
3346.875	3361.875	0.4462
3746.875	3766.250	0.5144
4135.625		
4175.625	4175.625	0.000
4207.500	4222.500	0.3552
4645.000	4666.250	0.4554
4966.875	4956.250	−0.2144
5056.250		
5106.250	5101.250	−0.0980
5148.750	5124.375	−0.4757
5540.000	5576.875	0.6612
5603.125	5633.125	0.5326
	5889.375	
6045.000	6002.500	−0.7080
6435.000	6450.000	0.2325
6490.000	6505.000	0.2306
6850.000	6872.500	0.3274
6955.000	6995.000	0.5718

TABLE IV. Comparison of Weyl estimate and cumulative frequency obtained experimentally after placing the scatterers.

Frequency (GHz)	Cumulative resonant frequency	Weyl estimate
1	0	0
2	0	0
3	3	0.9
4	6	4.5
5	11	9.7
6	17	16.3
7	22	24.5

amplitudes themselves may be different, as they depend on the location of the coupling, and hence to the way the modes are excited. Thus, we have obtained the energy spectrum for the given set of isospectral cavities and verified that each eigenvalue in one is present in the other at the same resonance value.

As a check on the quality of the spectral data, we compare the cumulative number of resonance levels as a function of frequency, obtained experimentally with the Weyl formula for the integrated density of states in a two-dimensional domain [11]:

$$N(k)=\frac{Ak^2}{4\pi}-\frac{Sk}{4\pi}+K, \tag{2}$$

where A and S are the area and perimeter of the domain, and K is a correction term associated with its topology. For a polygonal billiard with inner angles α_i, this is given by $K=\sum_i \frac{1}{24}(\pi/\alpha_i-\alpha_i/\pi)$. In the present case, we find $K=0.42$. We show in Table II a comparison between the experimental results with the above formula. The agreement is quite good.

Results for the chaotic geometry. We place the scatterers inside the cavity following the prescription outlined above. The scatterers are taken to be metallic cylinders of diameter 1.0 cm and height equal to the thickness of the cavities. The modified spectrum from the two cavities is shown in Fig. 5 and we list the first 22 resonances in Table III. We again find that the eigenvalues in the two cavities match to within 1%. Thus, there is clear evidence that isospectrality is retained in the modified chaotic geometry.

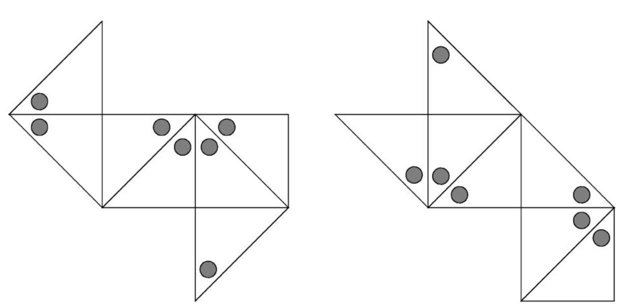

FIG. 6. A *nonisospectral* arrangement of scatterers. The scatterers in cavity $C2$ are now in a different position.

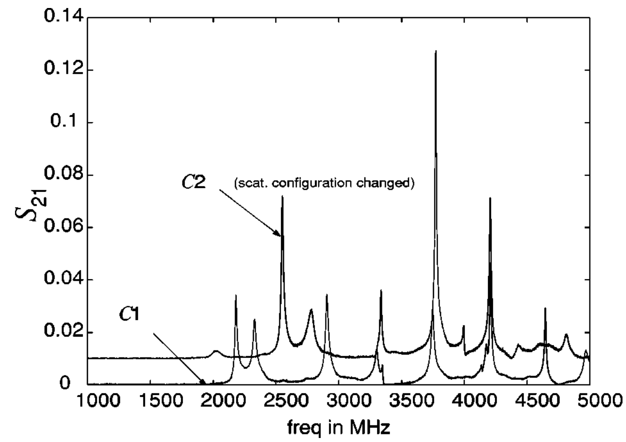

FIG. 7. Spectrum of the cavities in Fig. 6.

It may be noted that the introduction of the scatterers changes the topology of the domain from being simply connected to now being multiply connected. We now have a polygonal box with $p=7$ circular holes. In this case, the topology term in Weyl's formula for the integrated density of states is given by

$$K=\sum_i \frac{1}{24}\left(\frac{\pi}{\alpha_i}-\frac{\alpha_i}{\pi}\right)-\frac{p}{6}.$$

The comparison with Weyl's formula is given in Table IV. The agreement is not very good at low frequencies and the number of levels seems to be somewhat *higher* than that given by the Weyl estimate.

To make the demonstration more convincing and illustrate the nontriviality of the isospectral construction with scatterers, we consider another geometry (Fig. 6) where the scatterers in the second cavity are placed in a somewhat different manner. The arrangement still seems to follow the folding construction and naively one would expect isospectrality. However, on closer inspection, one finds that the correct correspondence between the edges of the subdomains has not been satisfied and the wave function matching condition in fact no longer holds and so we *should not* get isospectrality. We plot the spectrum for this case in Fig. 7. We see a marked difference from that in Fig. 5, namely, we find that there is no correspondence between the spectral lines from the two cavities. This shows clearly that isospectrality is indeed obtained only for the special arrangement of scatterers in Fig. 2.

In conclusion, we have demonstrated that isospectrality is unrelated to the underlying classical dynamics of a particle. We have shown a simple way of introducing scatterers of arbitrary shape into polygonal cavities in such a way that isospectrality is retained. This leads us to a new class of isospectral scatterers and also a better understanding of the essential features necessary for isospectrality.

We thank N. Kumar, Yashodhan Hatwalne, and Joseph Samuel for useful discussions. S.S. thanks the RRI for hospitality while this work was completed. A.D. acknowledges support from the NSF under Grant No. DMR 0086287. S.S. was partially supported by Grant No. NSF 0098801.

[1] C. Gordon, D. Webb, and S. Wolpert, Bull. Am. Math. Soc. **27**, 134 (1992).
[2] M. Kac, Am. Math. Monthly **73**, 1 (1966).
[3] P. Berard, Math. Ann. **292**, 547 (1992).
[4] S.J. Chapman, Am. Math. Monthly **102**, 124 (1995).
[5] S. Tabachnikov, *Billiards* (Sociéte Mathematique de France, Paris, 1995).
[6] H. J. Stockmann, *Quantum Chaos: An Introduction* (Cambridge Univeristy Press, Cambridge, 1999).
[7] S. Sridhar, D. Hogenboom, and B.A. Willemsen, J. Stat. Phys. **68**, 239 (1992).
[8] S. Sridhar and A. Kudrolli, Phys. Rev. Lett. **72**, 2175 (1994).
[9] V. Heuveline, J. Comput. Phys. **184**, 322 (2002).
[10] T.A. Driscoll, SIAM Rev. **39**, 1 (1997).
[11] H. P. Baltes and E. R. Hilf, *Spectra of Finite Systems* (Wissenschaftsverlag, Mannheim, 1976).